A WISE method for designing IIR filters.

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***Abstract*—The problem of designing optimal digital IIR filters with frequency responses approximating arbitrarily chosen complex functions is considered. The real-valued coefficients of the filter's transfer function are obtained by numerical minimization of carefully formulated cost, which is referred here to as the weighted integral of the squared error (WISE) criterion. The WISE criterion linearly combines the WLS criterion that is used in the weighted least squares approach toward filter design and some time-domain components. The WLS part of WISE enforces quality of the frequency response of the designed filter, while the time-domain part of the WISE criterion restricts the positions of the filter's poles to the interior of an origin-centred circle with arbitrary radius. This allows one not only to achieve stability of the filter but also to maintain some safety margins. A great advantage of the proposed approach is that it does not impose any constraints on the optimization problem and the optimal filter can be sought using off-the-shelf optimization procedures. The power of the proposed approach is illustrated with filter design examples that compare favorably with results published in research literature.**

***Index Terms*—Discrete time filters, IIR digital filters, least squares methods, optimization methods.**

NOMENCLATURE

$\hat{B}(z), \hat{A}(z)$	Numerator and denominator of the filter. These are real-coefficient polynomials in z^{-1}. $\hat{B}(z) = b_0 + \cdots + b_{n_b} z^{-n_b}$, $\hat{A}(z) = 1 + a_1 z^{-1} + \cdots + a_{n_a} z^{-n_a}$.
n_b, n_a	Degrees of polynomials $\hat{B}(z)$ and $\hat{A}(z)$, respectively.
$\boldsymbol{x}$	$\boldsymbol{x} = [a_1, \ldots, a_{n_a}, b_0, \ldots, b_{n_b}]^{\mathrm{T}}$ point in $n_a + n_b + 1$-dimensional space.
$\mathcal{P}$	$n_a + n_b + 1$-dimensional space of IIR filters.
$\mathcal{S}, \mathcal{S}_0, \mathcal{U}, \mathcal{B}, \mathcal{J}$	Various subspaces and subsets of $\mathcal{P}$.
$\overline{\mathcal{S}}$	Bar above the name of a subset denotes enclosure of the subset.
ρ	Admissible maximum radius of poles of the designed filter.
$C(\rho)$	Origin-centered circle on a complex plane with radius ρ.
$\hat{G}(z), \hat{F}(z), \hat{F}_2(z), \hat{F}_3(z)$	Rational functions in z^{-1}: $\hat{G}(z) \hat{=} \hat{B}(z)/\hat{A}(z)$, $\hat{F}(z) \hat{=} /\hat{A}(\rho z)$, $\hat{F}_3(z) \hat{=} \hat{F}^3(z)$
m	Number of distinct poles of $\hat{G}(z)$.
$p_1, p_2, \ldots, p_m$	Poles of $\hat{G}(z) \cdot \hat{A}(p_k) = 0$ for $k = 1, \ldots, m$.
$k_1, k_2, \ldots, k_m$	Multiplicities of poles of $\hat{G}(z)$. $\hat{A}(z) = \prod_{i=1}^{m} (1 - p_i z^{-1})^{k_i}$.
z^*	Star as a superscript denotes complex conjugation.
ν	Normalized frequency (sampling frequency is $\nu = 1$). All functions of ν are periodic $V(\nu) = V(\nu+1)$ and hermitean symmetric $V(-\nu) = V^*(\nu)$.
$A(\nu), B(\nu), G(\nu)$	Frequency response of $\hat{A}(z), \hat{B}(z)$ and $\hat{G}(z)$ respectively, e.g., $B(\nu) \hat{=} \hat{B}(e^{j2\pi\nu})$.
$H(\nu)$	Complex-valued target frequency response of the designed filter.
$E(\nu)$	Complex-valued frequency response error $E(\nu) \hat{=} G(\nu) - H(\nu)$
$W(\nu)$	Real, even, non-negatively valued frequency domain weight function.
t	Normalized time. Variable t is always a whole number.
$f(t), f_2(t), f_3(t)$	Impulse response of $\hat{F}(\rho z)$, $\hat{F}_2(\rho z)$, $\hat{F}_3(\rho z)$, respectively.
λ	Weighting factor—a real number such that $0 \leq \lambda \leq 1$.
J_{WLS}	Version of cost (optimization criteria).
J_{LEVY}	Version of cost (optimization criteria).
J_{WISE}	Version of cost (optimization criteria).
J_{Cheby}	Version of cost (optimization criteria).

I. INTRODUCTION

One of the most popular objectives of designing a filter is to shape the frequency response of the filter prototype in such a way that it approximates a pre-selected complex-valued function. In order to perform the design in an optimal manner, we

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choose a criterion (cost) that measures quality of the approximation. Very often, the cost takes a form of a squared weighted L_2 norm of the frequency response error

$$J_{\mathrm{WLS}}(\mathbf{x}) \hat{=} \int_{-0.5}^{0.5} |G(\nu) - H(\nu)|^2 W(\nu) d\nu = \int_{-0.5}^{0.5} E(\nu) E^*(\nu) W(\nu) d\nu \quad (1)$$

where

$G(\nu) = B(\nu)/A(\nu)$	frequency response of the designed filter;
$H(\nu)$	desired frequency response of the filter;
$E(\nu) \hat{=} G(\nu) - H(\nu)$	frequency response error,
$W(\nu)$	real, non-negative weight function.

The methods that aim at minimizing the above criterion traditionally share a common name: weighted least squares (WLS) methods. Consequently, cost (1) is called the WLS criterion.

Designing digital filters through minimization of (1) is a classic approach that has been successfully applied to constructing good quality FIR filters [1]. In such cases, the coefficients of the filter's transfer function are obtained by determining the minimum of a relatively simple quadratic form. Practical methods that are used for such calculations include an explicit matrix formula that expresses the solution in an analytic form [2] and an eigenfilter approach that is based on determining a suitable eigenvector of a square matrix [3].

Subsequent research has shown that the WLS approach can be used to construct FIR filters that are optimal in the sense of criteria other than those built around weighted L_2. Those new costs can be minimized by putting the WLS algorithm inside a loop that iteratively modifies the weight function $W(\nu)$. It has been demonstrated in [4] that by using the Lawson's algorithm (LA) for updating $W(\nu)$, it is possible to minimize weighted L_∞ (Chebyshev) norm of the frequency response error. Later, a modified LA was used [2] to achieve the same goal in much shorter time. LA constituted a competitive approach to the famous Remez exchange algorithm [5]. A great advantage of the approaches based on LA was that they could have been used for designing filters with arbitrarily chosen frequency responses, whereas the traditional version of the Remez exchange algorithm [5] was suitable only for the linear-phase ones. Only recently have L_∞ departures to nonlinear-phase become feasible, e.g., [6]. Burrus *et al.* [7] have broadened designers' options by proposing algorithms for modifying the weight function to enable designing filters that are optimal in the sense of criteria based on weighted L_p norm with arbitrary p.

These important achievements obtained for FIR filters are only partially matched by results of similar quality and strength pertaining to IIR filters. Design of IIR filters faces at least a couple of new problems that are hardly noticeable when FIR design is tackled. First, in the case of IIR filters, (1) is no longer a quadratic form of the transfer function's coefficients. Therefore, we have the following

a) Nonlinear optimization techniques have to be deployed to determine the coefficients of the filter transfer function for which (1) achieves minimum.

b) Cost (1) may have more than one local minimum. Consequently, we have no guarantee that the solution delivered by a numerical procedure that minimizes (1) is the global, and not a local, minimum. This is a common problem of virtually all general purpose optimization procedures, and we are not going to overcome this limitation in our paper.

Second, if no constraints are imposed on the optimization process, the filter, whose frequency response minimizes (1), can turn out to be unstable. Therefore, the formulation of the optimization problem must be extended beyond the bland requirement of minimizing (1).

As we mentioned earlier, design of IIR filters usually leads to usage of nonlinear optimization procedures. There is, however, one important exception that permits avoidance of this complication —Levy's linearization [8]. Although this method has some drawbacks, it has attracted a lot of attention and inspired a number of interesting algorithms. The main concept of Levy's linearization is to replace criterion (1) with

$$J_{\mathrm{LEVY}}(\mathbf{x}) \hat{=} \int_{-0.5}^{0.5} |B(\nu) - H(\nu)A(\nu)|^2 W_{\mathrm{LEVY}}(\nu) d\nu. \quad (2)$$

The new weight function $W_{\mathrm{LEVY}}(\nu)$ is defined as $W_{\mathrm{LEVY}}(\nu) \hat{=} W(\nu)/|A(\nu)|^2$, where $\hat{A}(\nu)$ is frequency response of optimal $\hat{A}(z)$. Levy's linearization simplifies the filter design problem. Cost (2) can be represented in a quadratic form similar to that obtained in design of FIR filters. Its minimization is similarly easy. However, Levy's linearization introduces a new problem. In order to properly formulate (2), one has to partially know the solution, i.e., the magnitude response of the denominator of the optimal filter transfer function $|\hat{A}(\nu)|$. A way of avoiding the problem is to apply an iterative Sanathanan–Koerner algorithm [8] that generates a sequence of improving approximations of the optimal $\hat{A}(\nu)$. Unfortunately, this algorithm may not converge, and even if it does converge, it may not reach a local minimum of cost (1) [12]. Another problem is that Levy's linearization does not address stability of the filter; hence, this aspect of the design has to be solved separately. Approaches that allow one to impose stability of the designed filter together with use of Levy's linearization can be found in [9]–[11]. Despite its apparent attractiveness, Levy's linearization shows rather limited practical usefulness. It is not a reliable tool for general purpose IIR filter design.

In order to minimize (1) without use of Levy's linearization, we may use one of the many general-purpose optimization techniques developed for solving nonlinear problems. A major issue that still has to be somehow tackled is stability of the filter.

It is perhaps, at this juncture, worth mentioning an area of optimization that has enjoyed some popularity throughout the past couple of decades: use of "natural algorithms." These algorithms forego traditional deterministic iteration schemes, seeking instead to somehow emulate processes observed in nature, such as Darwinian evolution and molecular alignment of cooling materials. This category of techniques includes such variants as genetic algorithms [13], Darwinian design/evolutionary programming, and simulated annealing [14], [15]. All of these methods have been applied to the design of fixed (and even adaptive) IIR filters in an attempt to overcome entrapment in local cost function minima. Although the issue of ensuring stability is usually easily achieved and some useful specific designs have emerged from several authors in the DSP literature, all these methods deploy numerous highly problem-specific parameters to aid convergence. We believe that the fragmentary successes obtained have provided little general insight and do not at present offer compelling alternatives to the more conventional thrust we advocate here.

Maintaining stability of an IIR filter whose coefficients are obtained by minimizing (1) is, in general, not straightforward. A literature search shows that many authors simply leave out this topic and concentrate merely on good approximation of the desired frequency response. Such an approach is not a major problem when one designs a filter for which only the magnitude, and not phase response, is important. In this case, all unstable factors in the filter transfer function $1/(1-p_i z^{-1})$ can be replaced with their stable counterparts that have identical magnitude response $1/(z^{-1} - p_i^*)$. If, however, phase response is also important, this method cannot be applied since it changes the phase response of the filter.

There are four major approaches that can be applied for filter stabilization when optimization techniques are used for minimizing (1). None of them is perfect, and hence, probably none of them can be recommended as a universal tool. Here, we give a brief overview of these approaches along with specimen references where more details can be found.

i) Historically, the first method was proposed by Deczky in the classic paper [16]. His concept was to modify a standard, gradient-based optimization. The author suggested that the starting point for the optimization procedure should be a stable filter. After that, the size of the searching step should be controlled so that the searching trajectory never leaves the stability area. Deczky's approach can be justified as follows. If the target frequency response $H(\nu)$ takes only finite values and any pole of the designed filter is placed on the unit circle, then cost (1) has infinite value. Therefore, any steepest decent trajectory that starts at the point representing a stable transfer function will avoid the border of stability.

   The algorithm proposed by Deczky is generally considered to be computationally expensive. However, its main drawback is the necessity to interfere with the contents of the optimization procedure, which may be difficult if one uses off-the-shelf general-purpose optimization software.

ii) The second approach to filter stabilization is to choose the target frequency response in such a way that the optimal filter is stable. Two examples of such an approach found in the literature suggested that if the target frequency response is minimum phase [17] or the group delay of the target frequency response is large enough [18], then the resultant filter will be stable. The problem is that often, the designer's flexibility to modify the target frequency response is restricted and, in many cases, may be too rigid to achieve stabilization of the filter.

iii) The third approach to filter stabilization is to impose explicit constraints on the coefficients of the denominator $\hat{A}(z)$ of the filter's transfer function to enforce stability of the filter. Since the expressions used for the inequality constraints complicate quickly when n_a increases, this approach is useful if $n_a \leq 2$. This limitation is enhanced by the fact that the space of vectors of coefficients that represent stable filters is convex only if $n_a = 1$ or $n_a = 2$. In all other cases, the space is concave. This seriously complicates the optimization process. The problem can be avoided when the denominator of the transfer function is represented as a product of at most second-order factors [9], [19]. In such a case, stability of the filter can be enforced by imposing linear inequality constraints on the coefficients of each factor. The drawback of factorization is that it creates saddle points in the optimization criterion. This may prevent the optimization algorithm from converging to a local minimum. An approach that eludes many of the problems mentioned above has been advanced in [11]. The authors avoided factorizing the denominator of the filter and maintained stability by imposing linear constraints: $\mathrm{Re}[A(\nu)] > 0$ on all $\nu \in [0, 0.5]$. Unfortunately, the constraints provide sufficient but not necessary conditions for filter stability. This means that some stable filters are permanently excluded from the set of admissible solutions, even if they are optimal in the sense of the preselected criterion. We will show an example where this limitation affects the quality of the filter.

iv) The fourth group of methods of imposing stability of the filter arise from modification of the cost in such a way that its minima always lie in the stable area. This approach has several advantages over the other ones mentioned above. First, unlike i), it does not require changing the code of the optimization procedure; secondly, unlike ii), this approach does not require changing the target frequency response; third, unlike iii), it allows use of unconstrained optimization methods that usually are much simpler and faster than those capable of handling constraints. An example of such an approach has been presented in [20]. The author represented the denominator of the transfer function as a product of first- and second-order factors and used nonlinear transformation of each factor so that regardless of the values of the optimization parameters, the transfer function was always stable.

In this paper, we propose an algorithm that also belongs to group iv). The novelty of the approach consists in formulation of a new cost—weighted integral of the squared magnitude response error (WISE criterion). This criterion consists of two components that are linearly combined. The first component is simply WLS performance index (1). The second component is chosen in such a way that as long as the poles of the designed filter stay inside the permissible area, the value of that component is practically constant. However, as soon as at least one pole leaves the designated area, the second component soars immediately, forcing the optimization algorithm to pull back and stay within the required limits. Here, the permissible area is defined as $C(\rho)$— a disk centred at the origin of the complex plane and radius ρ. The WISE optimization problem can be solved with use of any general-purpose numerical procedure capable of performing unconstrained minimization of nonlinear performance index. In order to ease implementation of the proposed approach, we show how to effectively calculate not only the value of the WISE criterion but also its gradient and Hessian. This supplementary information is useful when utilizing optimization platforms like the MATLAB environment. The results presented in this paper constitute direct expansion of our earlier work [21], which suggested its scope and promise.

II. Main Results

A. Confining Pole Positions of the Designed Filter—WISE Criterion

We start with defining some sets and spaces used throughout this section and in Appendix. Let $\mathcal{P}$ denote an $n_a + n_b + 1$-dimensional linear space of IIR filters whose transfer functions can be represented by

$$\hat{G}(z) = \frac{\hat{B}(z)}{\hat{A}(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{n_b} z^{-n_b}}{1 + a_1 z^{-1} + \cdots + a_{n_a} z^{-n_a}}. \quad (3)$$

Each point of the space is described by the vector of coefficients of the filter's transfer function $\mathbf{x} = [a_1, \ldots, a_{n_a}, b_0, \ldots, b_{n_b}]^{\mathrm{T}}$. The space consists of three subsets:

1) $\mathcal{S}$—representing filters with all poles inside $C(\rho)$;
2) $\mathcal{U}$—representing filters with at least one pole outside $C(\rho)$.
3) $\mathcal{B}$— separating $\mathcal{S}$ and $\mathcal{U}$ that represents filters with at least one pole at the border of $C(\rho)$ and with all other poles inside $C(\rho)$.

Set $\mathcal{S}$ contains an $n_b + 1$-dimensional linear subspace of $\mathcal{P}$, which is denoted $\mathcal{S}_0$, consisting of points representing filters with all poles placed at the origin (FIR filters).

Our filter design problem can be rephrased as follows. Obtain $\tilde{\mathbf{x}} \in \bar{\mathcal{S}}$ such that $J_{\mathrm{WLS}}(\tilde{\mathbf{x}}) \le J_{\mathrm{WLS}}(\mathbf{x})$ for every $\mathbf{x} \in \bar{\mathcal{S}}$. Here, $\bar{\mathcal{S}}$ denotes enclosure of $\mathcal{S}$. The solutions to this problem are called globally optimal. When we use a numerical procedure to minimize cost, e.g., (1), we usually solve a slightly different problem. Determine $\tilde{\mathbf{x}} \in \bar{\mathcal{S}}$ such that $J_{\mathrm{WLS}}(\tilde{\mathbf{x}}) \le J_{\mathrm{WLS}}(\mathbf{x})$ for every $\mathbf{x} \in \mathcal{S} \cap \mathcal{N}$, where $\mathcal{N}$ is some neighborhood of $\tilde{\mathbf{x}}$. Solutions to this problem are called locally optimal. Normally, different locally optimal solutions can be obtained by running the same optimization program using different parameters such as starting point (or points), initial length of the searching step, stop conditions, etc. This fact must be taken into account when filter design optimization techniques are compared with each other.

The method of confining pole positions that we propose to use in this paper is based on a technique that is occasionally deployed to solve some optimization problems with constraints. The approach consists of modifying the original cost (1) by combining it with auxiliary cost $J_{\mathrm{PEIR}}(\mathbf{x})$

$$J_{\mathrm{WISE}}(\mathbf{x}) = (1-\lambda) J_{\mathrm{WLS}}(\mathbf{x}) + \lambda J_{\mathrm{PEIR}}(\mathbf{x}) \quad (4)$$

where $\lambda \in [0, 1]$. Instead of performing minimization of cost (1) with constraints, we are going to minimize (4) without imposing any constraints. In order for both problems to have identical solutions, we should choose $J_{\mathrm{PEIR}}(\mathbf{x})$ and λ in such a way that the set of locally optimal solutions to the original problem is identical with the set of local minima of (4). Selecting such $J_{\mathrm{PEIR}}(\mathbf{x})$ and λ is not an easy task since we have to perform it without *a priori* knowledge of the positions of the locally optimal solutions. The best way of tackling the problem is to choose $J_{\mathrm{PEIR}}(\mathbf{x})$ such that $J_{\mathrm{PEIR}}(\mathbf{x})$ is (almost) constant inside $\mathcal{S}$, and it soars rapidly outside this area. This, with suitably chosen λ, will not only preserve the positions of cost's local minima inside $\bar{\mathcal{S}}$ but will also shape the steepest descent lines of $J_{\mathrm{WISE}}(\mathbf{x})$ in $\mathcal{U}$ so that their points of attraction are in $\mathcal{S}$.

Two demands that we impose on $J_{\mathrm{PEIR}}(\mathbf{x})$ are slightly contradicting. The gradient of $J_{\mathrm{PEIR}}(\mathbf{x})$ should be large outside $\mathcal{S}$ and close to zero inside it. Normally, the gradient is a continuous function of x, and it cannot be changed in a stepwise way. Therefore, one should expect existence of a "transition area" $\mathcal{J}$ in the close neighborhood of $\mathcal{B}$, where the gradient has intermediate values. A locally optimal solution placed in $\mathcal{B}$ may be affected by the existence of the transition area. The local minima of (4) appropriate to these solutions may not stay at the positions of the locally optimal solutions but be moved into their close neighborhood inside $\mathcal{J}$. Of course, the faster the gradient of $J_{\mathrm{PEIR}}(\mathbf{x})$ changes between $\mathcal{S}$ and $\mathcal{U}$, the narrower the transition area is, and the whole problem becomes less important. The following Lemma suggests how to construct $J_{\mathrm{PEIR}}(\mathbf{x})$ appropriate to our optimization problem.

Lemma 1: Polynomial $\hat{A}(z)$ has all its roots inside $C(\rho)$ if and only if the impulse response of $\hat{F}(z) = 1/\hat{A}(\rho z)$ satisfies $\lim_{l \to \infty} \mathrm{f}(t) = 0$.

Proof of this lemma is given in the Appendix.

In view of this Lemma, we propose to construct cost (4) as follows:

$$\begin{aligned} J_{\mathrm{WISE}}(\mathbf{x}) &\hat{=} (1-\lambda) \int_{-0.5}^{0.5} E(\nu) E^*(\nu) W(\nu) d\nu + \lambda \sum_{t=T+1}^{T+M} f^2(t) \\ &= (1-\lambda) J_{\mathrm{WLS}}(\mathbf{x}) + \lambda J_{\mathrm{PEIR}}(\mathbf{x}) \end{aligned} \quad (5)$$

where T and M are suitably chosen positive integers. The second term in (5) is referred to as partial energy of impulse response (PEIR). To distinguish WLS criterion (1) from (5), we will call the latter one the WISE criterion and the approach based on minimization of (5) the WISE method.

As we will see later, the properties of (5) are very much dependent on grad $J_{\mathrm{PEIR}}(\mathbf{X})$. This gradient can be calculated as follows. By definition

$$\text{grad } J_{\mathrm{PEIR}}(\mathbf{x}) = \left[\frac{\partial J_{\mathrm{PEIR}}}{\partial b_0}, \ldots, \frac{\partial J_{\mathrm{PEIR}}}{\partial b_{n_b}}, \frac{\partial J_{\mathrm{PEIR}}}{\partial a_1} \ldots, \frac{\partial J_{\mathrm{PEIR}}}{\partial a_{n_a}} \right]. \quad (6)$$

The derivatives of $J_{\mathrm{PEIR}}(\mathbf{x})$ with respect to b_i are zero. Therefore, we will only calculate derivatives of $J_{\mathrm{PEIR}}(\mathbf{x})$

with respect to a_i, $i = 1, \ldots, n_a$. It follows from (5) that $(\partial J_{\rm PEIR}/\partial a_i) = 2\sum_{t=T+1}^{T+M} f(t)(\partial f(t)/\partial a_i)$. Since

$$\begin{aligned}\frac{\partial}{\partial a_i} f(t) &= \frac{\partial}{\partial a_i}\int_{-0.5}^{0.5} \hat{F}(e^{j2\pi\nu})e^{j2\pi\nu t}d\nu \\ &= -\int_{-0.5}^{0.5} \frac{\rho^{-i}e^{j2\pi\nu(t-i)}}{\hat{A}^2(\rho e^{j2\pi\nu})}d\nu \\ &= -\rho^{-i} f_2(t-i) \end{aligned}\tag{7}$$

therefore

$$\frac{\partial J_{\rm PEIR}}{\partial a_i} = -2\rho^{-i}\sum_{t=T+1}^{T+M} f(t) f_2(t-i) \tag{8}$$

where $f_2(t)$ is the impulse response of $1/A^2(\rho z)$

Now, we have to decide how to choose the values of the WISE-specific parameters: T, M, and λ such that the gradients of the costs (1) and (5) are (almost) identical inside S and the steepest descent lines of (5) that start in $\mathcal{U}$ converge to $\mathcal{S}$. Although it is difficult, if possible at all, to give a precise formula that would produce the best values for those three numbers, it is still possible to provide practical guidelines that allow the designer to make a sensible choice of their values. Lemma 2 gives a few properties of $J_{\rm PEIR}(\mathbf{x})$, knowledge of which will help to formulate guidelines for selecting T, M, and λ

Let $||\mathbf{x}|| = \sqrt{\sum_{i=1}^{n_a} a_i^2}$ and $||{\rm grad}\ J(\mathbf{x})||_2 = \sqrt{\sum_{i=0}^{n_b}(\partial J(\mathbf{x})/\partial b_i)^2 + \sum_{i=1}^{n_a}(\partial J(\mathbf{x})/\partial a_i)^2}$. Note that $||\cdot||$ is a pseudo-norm since $||\mathbf{x}||$ can be zero even through $\mathbf{x} \neq \mathbf{0}$. On the other hand, $||\cdot||_2$ is the Euclidean norm (length) of its argument.

Lemma 2: If $M = n_a$, $T \geq n_a$, then

a) $J_{\rm PEIR}(\mathbf{x}) = 0$ if and only if $\mathbf{x} \in \mathcal{S}_0$;
b) $\lim_{||\mathbf{x}|| -> \infty} J_{\rm PEIR}(\mathbf{x}) = \infty$
c) grad $J_{\rm PEIR}(\mathbf{x}) = 0$ if and only if $\mathbf{x} \in \mathcal{S}_0$;
d) $\lim_{||\mathbf{x}|| -> \infty} ||\ {\rm grad}\ J_{\rm PEIR}(\mathbf{x})||_2 = \infty$;
e) if $\mathbf{x} \in \mathcal{S}$ then $\lim_{T -> \infty} ||{\rm grad}\ J_{\rm PEIR}(\mathbf{x})|| = 0$.

Proof of this lemma is given in the Appendix.

It follows from Lemma 2(c) that if $M = n_a$, $T \geq n_a$, and $\lambda = 1$, then $J_{\rm WISE}(\mathbf{x}) \neq \mathbf{0}$ for every $\mathbf{x} \in \bar{\mathcal{U}}$. Since grad $J_{\rm wise}(\mathbf{x})$ is a continuous function of λ therefore there exists $\varepsilon > 0$ such that for every $\mathbf{x} \in \bar{\mathcal{U}}$ grad $J_{\rm wise}(\mathbf{x}) \neq 0$ for $\lambda \in [1-\varepsilon, 1+\varepsilon]$. This implies that if $1 - \varepsilon < \lambda < 1$, then $J_{\rm wise}(\mathbf{x})$ has no local minima in $\bar{\mathcal{U}}$ or, in other words, all its local minima are located inside $\mathcal{S}$. In such a case, minimizing (5) yields a stable filter. In the reminder of this paper, we will assume that $M = n_a$, although $M > n_a$ can do the job as well.

Now, we need to make sure that $||{\rm grad}\ J_{\rm PEIR}(\mathbf{x})||_2$ is sufficiently small inside $\mathcal{S}$ so that grad $J_{\rm wise}(\mathbf{x})$ in this area is dominated by grad $J_{\rm WLS}(\mathbf{x})$ and very large in $\mathcal{U}$ so that grad $J_{\rm wise}(\mathbf{x})$ is dominated by $J_{\rm PEIR}(\mathbf{x})S$ outside $\mathcal{S}$. Thesis (e) of Lemma 2 suggests that we can achieve at least the first goal by selecting sufficiently large value of T. The discussion below will give us better understanding of how T affects the size of grad $J_{\rm PEIR}(\mathbf{x})$ and will help to choose its value in a rational way.

It can be easily checked that the impulse responses $f(t)$ and $f_2(t)$ used in (8) are given by $f(t) = \sum_{i=1}^{m} g_i(t)(p_i/\rho)^t$ and $f_2(t) = \sum_{i=1}^{m} h_i(t)(p_i/\rho)^t$, where $g_i(t)$ and $h_i(t)$ are polynomials of degrees $k_i - 1$ and $2k_i - 1$, respectively. It follows from (8) that $t \geq T$. In the subsequent discussion, we will use this fact to approximate grad $J_{\rm PEIR}(\mathbf{x})$ and its length with the dominant components. To simplify discussion, we assume that $|p_1| > |p_2|$ and $|p_2| \geq |p_3| \geq \cdots \geq |p_m|$, although similar results can be derived even when $|p_1| = |p_2|$. Note that

$$\begin{aligned} f(t) &= g_1(t)\left(\frac{p_1}{\rho}\right)^t \left(1 + \sum_{k=2}^{m}\frac{g_k(t)}{g_1(t)}\left(\frac{p_k}{p_1}\right)^t\right) \\ &= g_1(t)\left(\frac{p_1}{\rho}\right)^t \left(1 + o(t^{-1})\right) \end{aligned}\tag{9}$$

where $o(x)$ is a function such that $\lim_{x\to 0}(o(x)/x) = 0$. Let $g_1(t) = \alpha_{k_1-1}t^{k_1-1} + \cdots + \alpha_0$. Then

$$\begin{aligned} g_1(t) &= \alpha_{k_1-1}t^{k_1} - 1\left(1 + \frac{\alpha_{k_1-2}}{\alpha_{k_1-1}}t^{-1} + \cdots + \frac{\alpha_0}{\alpha_{k_1-1}}t^{-m_1}\right) \\ &= \alpha_{k_1-1}t^{k_1-1}\left(1 + o(t^{-0.5})\right). \end{aligned}\tag{10}$$

Hence

$$f(t) = \alpha_{k_1-1}t^{k_1-1}\left(\frac{p_1}{\rho}\right)^t \left(1 + o(t^{-1})\right)\left(1 + o\left(t^{-0.5}\right)\right). \tag{11}$$

In a similar way

$$f_2(t) = \beta_{2k_1-1}t^{2k_1-1}\left(\frac{p_1}{\rho}\right)^t \left(1 + o(t^{-1})\right)\left(1 + o\left(t^{-0.5}\right)\right). \tag{12}$$

Since $(t-i)^{2k_1-1} = t^{2k_1-1}(1 + o(t^{-0.5}))$, we can combine (8) with (11) and (12) to obtain

$$\begin{aligned}\frac{\partial J_{\rm PEIR}}{\partial a_i} &\approx -2\rho^{-i}\sum_{t=T+1}^{T+M}\alpha_{k_1-1}\beta_{2k_1-1}t^{3k_1-2}\left(\frac{p_1}{\rho}\right)^{2t-i} \\ &= \rho^{-i}\gamma\sum_{\tau=1}^{M}(T+\tau)^{3k_1-2}\left(\frac{p_1}{\rho}\right)^{2(T+\tau)-i}\end{aligned}\tag{13}$$

where $\gamma = -2\alpha_{k_1-1}\beta_{2k_1-1}$. By extracting the dominant components of (13), we obtain

$$\frac{\partial J_{\rm PEIR}}{\partial a_i} \approx \gamma T^{3k_1-2}\left(\frac{p_1}{\rho}\right)^{2T} p_1^{-i}\sum_{\tau=1}^{M}\left(\frac{p_1}{\rho}\right)^{2\tau}. \tag{14}$$

Now, we can approximate $||{\rm grad}\ J_{\rm PEIR}(\mathbf{x})||_2$ as

$$||{\rm grad}\ J_{\rm PEIR}(\mathbf{x})||_2 \approx |\gamma| T^{3k_1-2}\left(\frac{|p_1|}{\rho}\right)^{2T}\left|\sum_{\tau=1}^{M}\left(\frac{p_1}{\rho}\right)^{2\tau}\right|\sqrt{\sum_{i=1}^{M}|p_1^{-2}|^i}. \tag{15}$$

It is clearly visible from (15) that if $|p_1| < \rho$, then for sufficiently large values of T $||{\rm grad}\ J_{\rm PEIR}(\mathbf{x})||_2$ is small. On the other hand, when $|p_1| > \rho$, then $||{\rm grad}\ J_{\rm PEIR}(\mathbf{x})||_2$ is large. We propose to choose T in such a way that when the point $\mathbf{x}$ moves across the transition area $\mathcal{J}$, the gradient increases its length certain amount of times. Let the transition area be described by limits imposed on p_1 : $|p_1| \in [\rho_{\min}, \rho_{\max}]$, where $\rho_{\min} < \rho$

and $\rho_{\max} > \rho$. Let $\mathbf{x}_{\min}$ be the point in $\mathcal{P}$ appropriate to $|p_1| = \rho_{\min}$ and $\mathbf{x}_{\max}$ be the point appropriate to $|p_1| = \rho_{\max}$. Define $\eta \hat{=} (\|\text{grad } J_{\text{PEIR}}(\mathbf{x}_{\max})\|_2)/(\|\text{grad } J_{\text{PEIR}}(\mathbf{x}_{\min})\|_2)$. We will demand that η is sufficiently large. It follows from (15) that

$$\eta \approx \frac{\left|\sum_{\tau=1}^{M}\left(\frac{p_{1\max}}{\rho}\right)^{2\tau}\right|\sqrt{\sum_{i=1}^{M}\rho_{\max}^{-2i}}}{\left|\sum_{\tau=1}^{M}\left(\frac{p_{1\min}}{\rho}\right)^{2\tau}\right|\sqrt{\sum_{i=1}^{M}\rho_{\min}^{-2i}}}\left(\frac{\rho_{\max}}{\rho_{\min}}\right)^{2T} = \kappa\left(\frac{\rho_{\max}}{\rho_{\min}}\right)^{2T}. \tag{16}$$

Assuming that the transition area is very thin and, hence, $\mathbf{x}$ does not drift away much inside $\mathcal{J}$ when travelling from $\mathbf{x}_{\min}$ to $\mathbf{x}_{\max}$ so that $\mathbf{x}_{\min} \approx \mathbf{x}_{\max}$, we can claim that $p_{1\max} \approx p_{1\min}$ and $\rho_{\max} \approx \rho_{\min}$. In such a case, $\kappa \approx 1$, and

$$\eta \approx \left(\frac{\rho_{\max}}{\rho_{\min}}\right)^{2T}. \tag{17}$$

Therefore, if we demand that $\eta > 10^N$, then we should choose T such that

$$T > \frac{0.5N}{(\log_{10}(\rho_{\max}) - \log_{10}(\rho_{\min}))}. \tag{18}$$

It follows from (18) that one can narrow down the transition area $\mathcal{J}$ and/or enlarge the ratio η between the values of $J_{\text{PEIR}}(\mathbf{x})$ on both sides of $\mathcal{J}$ by increasing T. For example, if we demand that $\rho_{\max} = 1.005\rho$, $\rho_{\min} = 0.995\rho$ and $\eta > 10^N$, then we obtain $T > 115N$. However, if we select $\rho_{\max} = 1.003\rho$, $\rho_{\min} = 0.997\rho$, then the recommendation is $T > 192N$. The authors' experience shows that usually, it suffices to take $T \in [100, 500]$.

Even if the ratio between $\|\text{grad } J_{\text{PEIR}}(\mathbf{x})\|_2$ inside $\mathcal{U}$ and inside $\mathcal{S}$ is very large, we still have no guarantee that grad $J_{\text{WISE}}(\mathbf{x})$ in these areas is dominated by grad $J_{\text{PEIR}}(\mathbf{x})$ and grad $J_{\text{WLS}}(\mathbf{x})$, respectively. However, we can gain this confidence if λ is chosen so that $\lambda\|\text{grad } J_{\text{PEIR}}(\mathbf{x})\|$ and $(1-\lambda)\|\text{grad } J_{\text{WLS}}(\mathbf{x})\|$ are approximately equal in $\mathcal{B}$. Note that when $\mathbf{x} \in \mathcal{B}$, then $|p_1| = \rho$. It follows from (15) that in a typical case when $k_1 = 1$

$$\|\text{grad } \lambda J_{\text{PEIR}}(\mathbf{x})\|_2 \approx \lambda|\gamma|TM\sqrt{\sum_{i=1}^{M}\rho^{-2i}}. \tag{19}$$

Hence, the value of λ that matches the lengths of both weighted gradients is inverse proportional to T. Our experience shows that for the values of T recommended in this paper, a sensible choice of λ is in the range $[10^{-10}, 10^{-3}]$. It is worth noting that the larger T is the less sensitive is the resultant filter to the value of λ. This fact is clearly illustrated in Example 3.

B. Gradient and Hessian of the Cost Function

Cost (5) can be minimized with use of practically any general-purpose numerical algorithm capable of solving multidimensional, nonlinear optimization problem without constraints. Normally, such algorithms require access to a subroutine calculating the cost to be minimized. More sophisticated and usually more efficient procedures may also need access to the gradient and even the Hessian of the cost [22]. Therefore, we provide here analytical formulation of these derivatives and formulate practical guidelines on how to calculate them numerically.

The gradient of the WISE cost can be obtained analytically by using

$$\frac{\partial J_{\text{WISE}}(\mathbf{x})}{\partial b_k} = 2(1-\lambda)\int_{-0.5}^{0.5}\frac{E(\nu)W(\nu)}{A^*(\nu)} \times e^{j2\pi\nu k}d\nu \quad 0 \le k < n_b, \tag{20}$$

$$\frac{\partial J_{\text{WISE}}(\mathbf{x})}{\partial a_l} = -2(1-\lambda)\int_{-0.5}^{0.5}\frac{E(\nu)G^*(\nu)}{A^*(\nu)^2}W(\nu)e^{j2\pi\nu l}d\nu - 2\lambda\rho^{-l}\sum_{t=T}^{T+M}f(t)f_2(t-l) \quad 0 < l < n_a. \tag{21}$$

The Hessian of $J_{\text{WISE}}(x)$ can be obtained from

$$\frac{\partial^2 J_{\text{WISE}}(\mathbf{x})}{\partial b_k \partial b_l} = 2(1-\lambda)\int_{-0.5}^{0.5}\frac{W(\nu)}{|A(\nu)|^2} \times \exp\left[j2\pi\nu(k-l)\right]d\nu \quad 0 \le k, l < n_b \tag{22}$$

$$\frac{\partial^2 J_{\text{WISE}}(\mathbf{x})}{\partial b_k \partial a_l} = -2(1-\lambda) \times \left(\int_{-0.5}^{0.5}\frac{G(\nu)W(\nu)}{|A(\nu)|^2}\exp\left[j2\pi\nu(k-l)\right]d\nu + \int_{-0.5}^{0.5}\frac{E(\nu)W(\nu)}{A^*(\nu)^2} \times \exp\left[j2\pi\nu(k+l)\right]d\nu\right) \quad 0 \le k < n_b,\ 0 < l < n_a, \tag{23}$$

$$\frac{\partial^2 J_{\text{WISE}}(\mathbf{x})}{\partial a_k \partial a_l} = (1-\lambda)\left(2\int_{-0.5}^{0.5}\left|\frac{G(\nu)}{A(\nu)}\right|^2 W(\nu)e^{j2\pi\nu(k-l)}d\nu + 4\int_{-0.5}^{0.5}\frac{E(\nu)G^*(\nu)}{A^*(\nu)^2}W(\nu) \times \exp\left[j2\pi\nu(k+l)\right]d\nu\right) + \lambda\rho^{-k-l}\sum_{t=T}^{T+M}(2f_2(t-k)f_2(t-l) + 4f(t)f_3(t-k-l)) \quad 0 < k,\ l < n_a \tag{24}$$

where $f_3(t)$ is the impulse response of $1/A^3(\rho z)$.

C. Practical Aspects of Calculating the Cost and Its Derivatives

It follows from (5) and (20)–(24) that every time we need to find out the values of the cost $J_{\text{WISE}}(\mathbf{x})$ and its derivatives, we have to calculate a few integrals. In a few benign cases, the expressions will have forms permitting closed-form integration to be done, but in most situations, we propose to use approximation of the integrals with finite sums. Since the function integrated in (5) is periodic, we can approximate the integral in that expression as follows:

$$\int_{-0.5}^{0.5} E(\nu)E^*(\nu)W(\nu)d\nu = \int_{0}^{1} E(\nu)E^*(\nu)W(\nu)d\nu \approx \frac{1}{N}\sum_{n=0}^{N-1} E(\frac{n}{N})E^*(\frac{n}{N})W(\frac{n}{N}). \quad (25)$$

In a similar way, we can tackle integrals in (20)–(24). For example, integrals in (20) and (22) can be approximated with

$$\int_{-0.5}^{0.5} \frac{E(\nu)W(\nu)}{A^*(\nu)} e^{j2\pi\nu k} d\nu \approx \frac{1}{N}\sum_{n=0}^{N-1} \frac{E\left(\frac{n}{N}\right)W\left(\frac{n}{N}\right)}{A^*\left(\frac{n}{N}\right)} e^{j2\pi\frac{kn}{N}} \quad (26)$$

$$\int_{-0.5}^{0.5} \frac{W(\nu)}{|A(\nu)|^2} \exp[j2\pi\nu(k-l)]d\nu \approx \frac{1}{N}\sum_{n=0}^{N-1} \frac{W\left(\frac{n}{N}\right)}{|A\left(\frac{n}{N}\right)|^2} \times \exp\left[j2\pi\frac{n(k-l)}{N}\right] \hat{=} \tilde{I}_{bb}(k-l). \quad (27)$$

Note that all summations that replace integrals in (20)–(24) are in fact inverse discrete fourier transforms (IDFTs) of appropriate spectra. Time and computational effort can be saved by deploying here the inverse FFT algorithm, provided that N exceeds sufficiently n_a+n_b. The only problem with this approach is that the argument $k-l$ that appears in (22)–(24) can go negative. When inverse FFT is used in formulas similar to (27), the argument is allowed to vary only between 0 and $N-1$. This limitation can be overcome by exploiting periodicity of the IDFT, e.g., $\tilde{I}_{bb}(k-l) = \tilde{I}_{bb}[(k-l) \bmod N]$.

It is worth mentioning that the approximated gradient and Hessian calculated with use of formulas similar to (26) and (27) are in fact the accurate gradient and Hessian of the approximated cost function calculated with use of (25). Therefore, when approximations like (25) and (27) are applied for calculating the cost, its gradient, and the Hessian, the original problem of minimizing (5) is practically replaced with the demand of minimizing a summed version of the WISE criterion

$$\tilde{J}_{\text{WISE}}(\mathbf{x}) = \frac{1-\lambda}{N}\sum_{n=0}^{N-1} E\left(\frac{n}{N}\right)E^*\left(\frac{n}{N}\right)W\left(\frac{n}{N}\right) + \lambda\sum_{t=T}^{T+M} f(t)^2. \quad (28)$$

III. Numerical Examples

In this section, we present examples of designing digital filters using WISE approach. The method has been implemented in MATLAB with use of the Optimization Toolbox, version 2. To minimize cost (28), we have used function *fminunc*[22], which is capable of solving nonlinear minimization problems without constraints. The function utilizes both the gradient and the Hessian of the cost.

A. Example 1

In the first example, we design a high-pass filter whose specifications were proposed in [11]. The required frequency response is

$$H(\nu) = \begin{cases} e^{-j24\pi\nu}, & \text{if } |\nu| > 0.25 \\ 0, & \text{if } |\nu| < 0.25 \end{cases} \quad (29)$$

and the size of the filter is described by $n_a = n_b = 14$. To allow a transition band in the filter's response, we introduce a weight function in a similar way to what was done in [11]: $W(\nu) = 0$ if $|\nu| \in [0.2375, 0.2625]$; otherwise, $W(\nu) = 1$. In order to stabilize the filter, we chose $\rho = 1$ and created the following WISE criterion:

$$J_{\text{WLS}} = 0.9999\int_{-0.5}^{0.5} |G(\nu)-H(\nu)|^2 W(\nu)d\nu + 0.0001\sum_{t=500}^{513} f(t)^2. \quad (30)$$

The results are compared to those presented in [11]. Fig. 1 shows the plots of magnitude of weighted frequency response errors for both filters. Note that the results obtained by WISE method compare very favorably with the outcome of [11]. The reason for the difference in the quality of the filters is that WISE method allowed the search for solution throughout the whole space of stable filters, whereas the approach in [11] restricted the search to a subset described $\text{Re}[A(\nu)] > 0$; see iii) in Section III of this paper. Fig. 2 shows the plots of $\text{Re}[A(\nu)]$ for both filters. It is obvious from these plots that filter that we designed was excluded from the set of admissible solutions in [11].

B. Example 2

This example shows how the WISE method can be utilized to design filters that are optimal in the sense of weighted Chebyshev norm. This objective is achieved with use of a modified Lawson's algorithm. We follow, to some extend, the concept that was successfully applied for FIR filters [2]. The basic idea of the approach is to solve a sequence of WLS (in FIR case) or WISE (in IIR case) problems with the weight function being modified as follows:

$$W_k(\nu) = W_{k-1}(\nu)\text{envel}\left(|W_0(\nu)E_{k-1}(\nu)|\right). \quad (31)$$

Here, $W_0(\nu)$ denotes the weight for the Chebyshev norm, whereas envel $(|W_0(\nu)E_{k-1}(\nu)|)$ is the envelope of the magnitude of the weighted frequency response error. The objective is to design a filter that minimizes $\max|W_0(\nu)E(\nu)|$. The

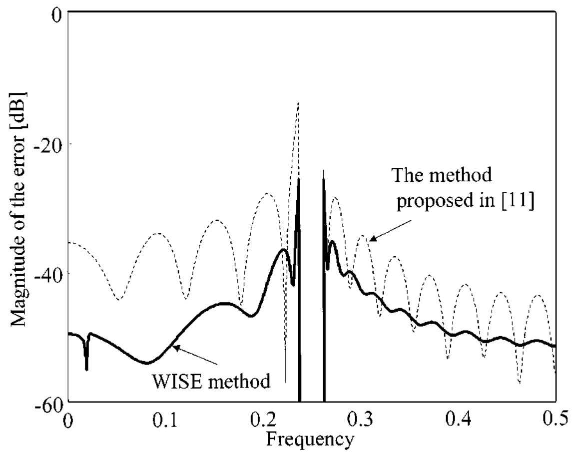

Fig. 1. Magnitude of frequency response error for both filters.

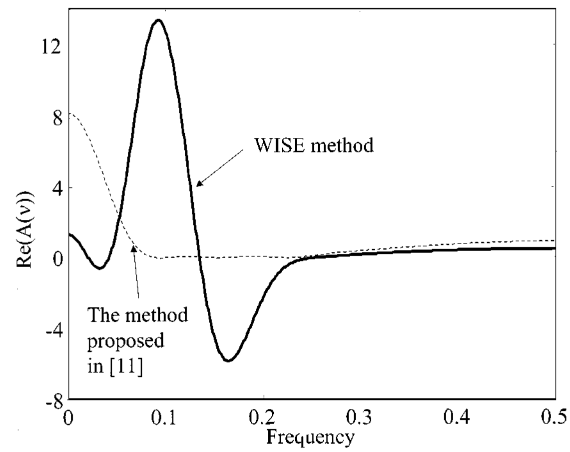

Fig. 2. Comparison of for both filters.

filter that we attempted here was a two-band filter whose ideal frequency response was given by

$$H(\nu) = \begin{cases} \exp(-j2\pi\nu 14.3), & \text{when } |\nu| < 0.25 \\ 0.5\exp(-j2\pi\nu 20), & \text{otherwise.} \end{cases} \quad (32)$$

Note that the required delay of the filter is 14.3 samples in the low-frequency band and 20 samples in the high-frequency band. Moreover, the required magnitude response of the filter drops down from unity in the low-frequency band to 0.5 in the high frequencies. We allow a narrow transition band by using the following weight function for the Chebyshev norm:

$$W_0(\nu) = \begin{cases} 0, & \text{when } 0.23 < |\nu| < 0.27 \\ 1, & \text{otherwise.} \end{cases} \quad (33)$$

An additional requirement is that the maximum magnitude of the poles of the designed filter should not exceed $\rho = 0.95$. Here, we will design a family of optimal filters, each of them having 31 coefficients to tune, i.e., $n_a + n_b + 1 = 31$. We start with FIR filter $(n_a = 0)$ and then gradually increase this number, ending up with so-called all-pole structure $(n_a = 30, n_b = 0)$. For the needs of the PEIR part of the criterion, we always use $T = 300$, $M = n_a$, and $\lambda = 0.000\,01$. Table I shows the value of the weighted Chebyshev norm obtained for filters with $n_a \in [0, 15]$. Notice that the best IIR filter was obtained when $n_a = 2$. Fig. 3 shows the weighted magnitude of the frequency response error for FIR filter and for the best IIR filter. Finally, Fig. 4 shows phase delays and Fig. 5 magnitude responses of these filters. Note that the IIR filter is significantly better than its FIR counterpart.

C. Example 3

This example illustrates how sensitive the WISE optimal filters are to the choice of T and λ. We analyze two features of the filters: their quality (as measured by WLS cost) and the largest magnitude of the filter poles.

The target frequency response is the same as in Example 2. The permissible radius for the filter poles is again $\rho = 0.95$. The structure of the filter is defined $n_a = 4$ and $n_b = 26$. In this case, our objective is to design a filter that minimizes WLS cost rather

TABLE I
VALUE OF THE WEIGHTED CHEBYSHEV NORM AS A FUNCTION OF THE NUMBER OF NONZERO POLES OF THE FILTER

n_a	J_{Cheby}	n_a	J_{Cheby}	n_a	J_{Cheby}	n_a	J_{Cheby}
0	0.2605	4	0.1211	8	0.1452	12	0.1611
1	0.2625	5	0.1236	9	0.1191	13	0.5744
2	0.1133	6	0.1180	10	0.1859	14	0.5590
3	0.1141	7	0.1182	11	0.1755	15	0.7232

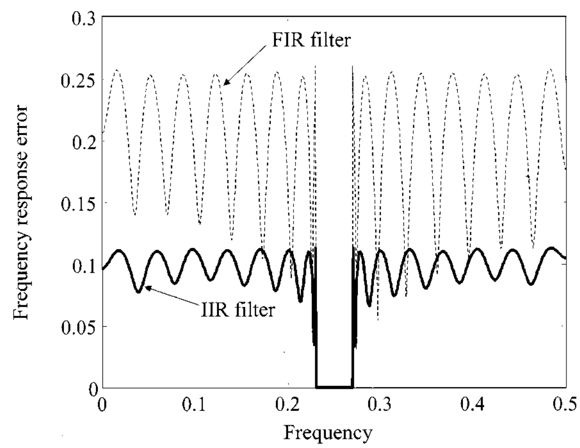

Fig. 3. Frequency response error of FIR and two-pole IIR filter.

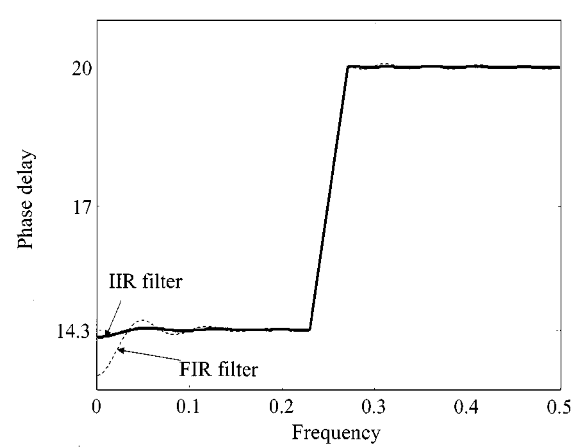

Fig. 4. Comparison of phase delays of FIR and IIR filters.

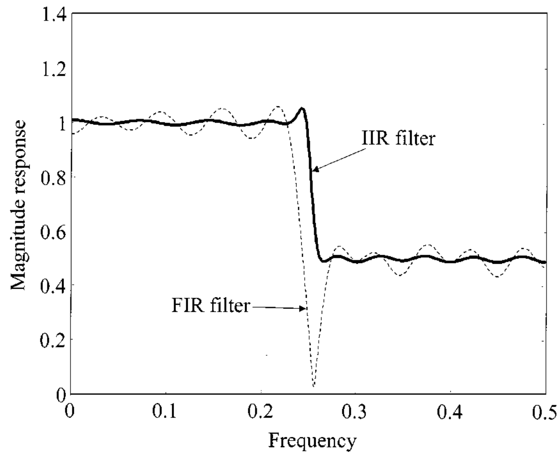

Fig. 5. Magnitude responses of FIR and two-pole IIR filters.

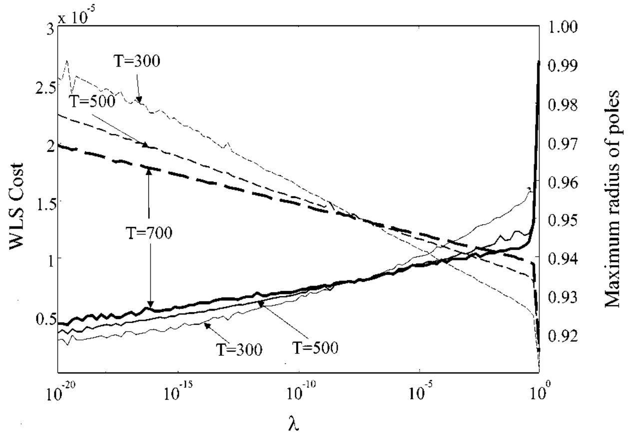

Fig. 6. Sensitivity of WISE-optimal filters to values of T and λ. Solid lines: WLS cost. Dashed lines: maximum radius of poles.

than the Chebyshev norm of the weighted error. Function (39) is used as the weight in WLS part of the WISE criterion. We set $M = 4$. The other two WISE-specific parameters are varied. T takes three different values (300, 500, and 700), whereas λ changes on a logarithmic scale between 10^{-20} and 0.99. For each pair (T, λ), we design a WISE optimal filter and record its quality, as measured by WLS criterion, and the largest radius of the filter poles. The results are shown in Fig. 6. Note that both analyzed features of the filter are not very sensitive to the choice of λ. Moreover, the sensitivity decreases when T goes larger. Note that in this example, some minimum value of λ must be maintained to confine the poles to the interior of $C(\rho)$. If λ is increased above this value, the poles stay inside the permissible area, but the quality of the filter slightly deteriorates. Significant deterioration occurs when λ approaches 1.

Analysis of the results in this example suggests a simple method of verifying whether or not the value of λ was correctly chosen. First, if for $\lambda = 0$ all poles of the filter are inside $C(\rho)$, then WISE design reduces to the WLS approach. Otherwise, λ should be increased until the largest magnitude of the filter poles equals ρ. It is also possible to check whether or not the value of T is properly chosen. If the quality of the filter designed with λ being tuned as described above does not change significantly when T increases, it means that T is sufficiently large, and the gradient of J_{PEIR} is small enough to allow the local minima of J_{WISE} to stay at the positions of local minima of J_{WLS} inside $\mathcal{S}$. Such subtle tuning of T and λ is needed only when the designer needs to approach the optimal solution very closely. Otherwise, one or two iterations of choosing T and λ suffice to obtain a satisfactory filter.

IV. Conclusions

A new approach to designing IIR filters whose frequency responses approximate an arbitrarily chosen complex-valued function has been presented. The method allows control of pole positions of the designed filter by confining them to the interior of an origin-centred circle with selectable radius. The novelty of the approach consists in such reformulation of the problem that it can be solved as an optimization problem without constraints. This is achieved by expanding the popular WLS criterion to embrace another time-domain term—PEIR—which, if properly used, preserves local minima of the original cost inside the permissible area and removes all local minima outside it. The new criterion is called the weighted integral of the squared error (WISE). It is worth mentioning that if the WISE specific parameters are selected as recommended in this paper, then in the case of FIR design, the WISE criterion automatically collapses to WLS. Therefore, the new criterion can be considered to be a natural and seamless expansion of WLS. The gradient and the Hessian of the WISE cost function have been obtained to allow implementation of the method by using virtually all general-purpose optimization packages. The proposed approach has been tested on a number of design tasks and proved to be robust, efficient, and fast in delivering good quality results.

Appendix

A. Proof of Lemma 1

Necessity ($\Rightarrow$)*:* If $\hat{A}(z)$ has all its roots inside $C(\rho)$, then $\hat{F}(z)$ is stable, and its impulse response satisfies$\sum_{t=0}^{\infty} f(t)| < \infty$. This implies that $\lim_{t\to\infty} f(t) = 0$.

Sufficiency ($\Leftarrow$)*:* □

Let $f_m(t)$ be the impulse response of $\hat{F}_m(z) = z^{-n_a}/\hat{A}(\rho z)$. Since

$$f(t) = f_m(t + n_a) \tag{A1}$$

it suffices to prove that if $\lim_{t\to\infty} f_m(t) = 0$, then polynomial $\hat{A}(z)$ has all its roots inside $\mathcal{C}(\rho)$. We can model $\hat{F}_m(z)$ as a cascade of n_a first-order state space subsystems whose transfer functions are $z^{-1}/(1-p_i(\rho z)^{-1})$. Let the output of the ith subsystem is the input to the $i+1$st subsystem, and let the numbers p_i satisfy $|p_1| \geq |p_2| \geq \cdots \geq |p_{n_a}|$. By denoting the input, output, and the state of each subsystem as u_i, y_i, and s_i, respectively, we can build their state-space models:

$$s_i(t+1) = p_i\rho^{-1}s_i(t) + u_i(t), \text{ for } i = 1, \ldots, n_a \tag{A2}$$
$$y_i(t) = s_i(t), \text{ for } i = 1, \ldots, n_a. \tag{A3}$$

These can be combined into one state-space model of $F_m(z)$ with $u(t) = u_1(t)$ being the input and $y(t) = y_{n_a}(t)$ being the output signal

$$\mathbf{s}(t+1) = \mathbf{A}\mathbf{s}(t) + \mathbf{B}u(t) \tag{A4}$$
$$y(t) = \mathbf{C}\mathbf{s}(t) \tag{A5}$$

where

$$\mathbf{A} \hat{=} \begin{bmatrix} p_1\rho^{-1}, & 0, & \cdots, & 0, & 0 \\ 1, & p_2\rho^{-1}, & \cdots, & 0, & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0, & 0, & \cdots, & 1, & p_{n_a}\rho^{-1} \end{bmatrix}, \mathbf{B} \hat{=} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathbf{C} \hat{=} [0, \ldots, 0, 1] \text{ and } s(t) \hat{=} \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_{n_a}(t) \end{bmatrix}.$$

It follows from (A4) and (A5) that $f_m(t) = \mathbf{C}\mathbf{A}^t\mathbf{B}$. If $\lim_{t\to\infty} f_m(t) = 0$, then $\lim_{t\to\infty} \mathbf{f}_m(t) = 0$, where

$$\mathbf{f}_m(t) = [f_m(t), f_m(t+1), \ldots, f_m(t+n_a-1)]_{\mathrm{T}}. \tag{A6}$$

Vector $\mathbf{f}_m(t)$ can be expressed as $\mathbf{f}_m(t) = \mathbf{O}_{\mathrm{M}}\mathbf{s}(t)$, where the observability matrix $\mathbf{O}_{\mathrm{M}}$ is given by $\mathbf{O}_{\mathrm{M}} = [(\mathbf{A}^{\mathrm{T}})^0\mathbf{C}^{\mathrm{T}}, \ldots, (\mathbf{A}^{\mathrm{T}})^{n_a-1}\mathbf{C}^{\mathrm{T}}]^{\mathrm{T}}$. Since $\det\mathbf{O}_{\mathrm{M}} = 1$, the observability matrix is not singular. Hence

$$\mathbf{s}(t) = \mathbf{O}_{\mathrm{M}}^{-1}\mathbf{f}_m(t) \tag{A7}$$

and $\lim_{t\to\infty} s(t) = 0$. Consequently, $\lim_{t\to\infty} s_1(t) = 0$. By using (A2), we obtain $s_1(t) = (p_1\rho^{-1})^t$. Therefore, $\rho > |p_1|$. Since all other numbers p_i have magnitude no larger than p_1, we conclude that all of them are inside $\mathcal{C}(\rho)$. □

B. Proof of Lemma 2

Thesis (a): Necessity ($\Rightarrow$): Note that $J_{\mathrm{PEIR}} = 0$ implies $f(t) = 0$, $t = T+1, T+2, \ldots, T+n_a$. Using notation introduced in the proof of Lemma 1, we can describe this observation as $\mathbf{f}_m(T+1+n_a) = 0$. It follows from (A7) that $\mathbf{s}(T+1+n_a) = 0$, and consequently, $s_1(T+1+n_a) = 0$. Since $s_1(T+1+n_a) = (p_1\rho^{-1})^{T+1+n_a}$, we conclude that $p_1 = 0$. This implies that $p_i = 0$ for $i = 1, 2, \ldots m$

Thesis (a): Sufficiency($\Leftarrow$): If $x \in S_0$, then $\hat{F}(z) = 1$. Therefore, $f(t) = 0$ for $t > 0$. This implies that $J_{\mathrm{PEIR}} = 0$.

Thesis (b): Let $\bar{\mathbf{x}} = [\bar{a}_1, \ldots, \bar{a}_{n_a}, \bar{b}_0, \ldots, \bar{b}_{n_b}] \in \mathcal{P}$ be such that $\|\bar{\mathbf{x}}\| = 1$. Denote the poles of the filter represented by $\bar{\mathbf{x}}$ as $\bar{p}_1, \ldots, \bar{p}_m$. Let $X(\alpha) \in P$ represent a filter with zeros identical with zeros of $\bar{\mathbf{x}}$ and poles $[\alpha\bar{p}_1, \ldots, \alpha\bar{p}_m]$. It can be easily verified that $\lim_{\alpha\to\infty} \|x(\alpha)\| \to \infty$. Let $\hat{\bar{F}}(z) = \prod_{i=1}^{m}(1-\bar{p}_k\rho^{-1}z^{-1})^{-k_i}$ and $\hat{F}_\alpha(z) = \prod_{i=1}^{m}(1-(\alpha\bar{p}_k)\rho^{-1}z^{-1})_i^{-k}$. Note that $\hat{F}_\alpha(z) = \hat{F}(\alpha^{-1}z)$. By using partial fraction expansion, we can represent $\hat{\bar{F}}(z)$ as

$$\hat{\bar{F}}(z) = \sum_{i=1}^{m} \frac{\sum_{l=0}^{k_i} c_{li}z^{-l}}{(1-\bar{p}_k\rho^{-1}z^{-1})^{k_i}}. \tag{A8}$$

Therefore

$$\hat{F}_\alpha(z) = \sum_{i=1}^{m} \frac{\sum_{l=0}^{k_i} c_{li}\alpha^l z^{-l}}{(1-\bar{p}_k\rho^{-1}\alpha z^{-1})^{k_i}}. \tag{A9}$$

The impulse response of $F_\alpha(z)$ is $f_\alpha(t) = \sum_{i=1}^{m}\sum_{l=0}^{k_i} c_{li}\alpha^l(\alpha\bar{p}_i\rho^{-1})^{t-l}v_{k_i}(t-l)$, where $v_{k_i}(t)$ are polynomials whose coefficients depend only on the multiplicity of the appropriate pole, e.g., $v_1(t) = 1$, $v_2(t) = t+1$, $v_3(t) = 0.5t^2 + 1.5t + 1$, etc. Now, we obtain

$$f_\alpha(t) = \alpha^t \sum_{i=1}^{m}\sum_{l=0}^{k_i} c_{li}\left(\bar{p}_i\rho^{-1}\right)^{t-l} v_{k_i}(t-l) = \alpha^t\bar{f}(t). \tag{A10}$$

It follows from (A10) and (5) that

$$J_{\mathrm{PEIR}}(\mathbf{x}(\alpha)) = \sum_{t=T+1}^{T+M} \alpha^{2t}\bar{f}^2(t). \tag{A11}$$

Let us impose an additional constraint on $\bar{\mathbf{x}}$, namely, that $J_{\mathrm{PEIR}}(\bar{\mathbf{x}}) \leq J_{\mathrm{PEIR}}(\mathbf{x})$ for all $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$. Now, if we choose $\mathbf{x}$ such that $\|\mathbf{x}\| = \|\mathbf{x}(\alpha)\|$, then $J_{\mathrm{PEIR}}(\mathbf{x}(\alpha)) \leq J_{\mathrm{PEIR}}(\mathbf{x})$. Since $\bar{\mathbf{x}} \notin S_0$ and therefore $J_{\mathrm{PEIR}}(\bar{\mathbf{x}}) = \sum_{t=T+1}^{T+M} f^2(t) \neq 0$, we see that $J_{\mathrm{PEIR}}(\mathbf{x}(\alpha))$ is a polynomial in α with non-negative coefficients such that not all of them are zero. Hence, $\lim_{\alpha\to\infty} J_{\mathrm{PEIR}}(\mathbf{x}(\alpha)) = \infty$. This implies that $\lim_{\|\mathbf{x}\|\to\infty} J_{\mathrm{PEIR}}(\mathbf{x}) = \infty$

Thesis (c): Necessity ($\Rightarrow$): The initial $n_b + 1$ elements of grad $J_{\mathrm{PEIR}}(\mathbf{x})$ are always zero. Therefore, it suffices to prove that when grad $_A J_{\mathrm{PEIR}}(\mathbf{x}) = 0$, then $\mathbf{x} \in \mathcal{S}_0$, where

$$\mathrm{grad}\;_A J_{\mathrm{PEIR}}(\mathbf{x}) \hat{=} \left[\frac{\partial J_{\mathrm{PEIR}}(\mathbf{x})}{\partial a_1}, \quad \ldots, \quad \frac{\partial J_{\mathrm{PEIR}}(\mathbf{x})}{\partial a_{n_a}}\right]. \tag{A12}$$

Since $M = n_a$, it follows from (8) that

$$\begin{aligned}\mathrm{grad}_A J_{\mathrm{PEIR}}(\mathbf{x}) = & -2[f(T+n_a) \\ & \times f(T+n_a-1), \ldots, f(T+1)]\mathbf{H}_2\mathbf{R}\end{aligned} \tag{A13}$$

where $\mathbf{H}_2$ is $n_a \times n_a$ Hankel matrix shown in (A14) at the top of the next page, and $\mathbf{R}$ is a diagonal matrix

$$\mathbf{R} = \begin{bmatrix} \rho^{-1}, & 0, & \ldots, & 0 \\ 0, & \rho^{-2}, & \ldots, & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & \ldots, & \rho^{-n_a} \end{bmatrix}. \tag{A15}$$

Since the length of one side of $\mathbf{H}_2(n_a)$ is not greater than the order of the system whose impulse response was used to gen-

$$\mathbf{H}_2 = \begin{bmatrix} f_2(T+n_a-1), & f_2(T+n_a-2), & \dots, & f_2(T) \\ f_2(T+n_a-2), & f_2(T+n_a-3), & \dots, & f_2(T-1) \\ \vdots & \vdots & \ddots & \vdots \\ f_2(T), & f_2(T-1), & \dots, & f_2(T-n_a+1) \end{bmatrix} \tag{A14}$$

$$\|\text{grad}_A J_{\text{PEIR}}(\mathbf{x})\|_2 = 2\sqrt{[a_1, \quad a_2, \quad \dots, \quad a_{n_a}]\,\mathbf{H}\mathbf{H}_2\mathbf{R}\mathbf{R}^{\text{T}}\mathbf{H}_2^{\text{T}}\mathbf{H}^{\text{T}}[a_1, \quad a_2, \quad \dots, \quad a_{n_a}]^{\text{T}}} \tag{A18}$$

erate the matrix ($\hat{F}_2(z) = 1/\hat{A}^2(\rho z)$ is of order $2n_a$), then $\det(\mathbf{H}_2) \neq 0$ [23].

Now, note that the impulse response of $\hat{F}(z) = 1/\hat{A}(\rho z)$ can be generated using recursive formula $f(t) = -\sum_{i=1}^{n_a} \rho^{-1} a_i f(t-i)$. Therefore, (A13) can be put as

$$\text{grad}_A J_{\text{PEIR}}(\mathbf{x}) = -2[a_1, \quad a_1, \quad \dots, \quad a_{n_a}]\mathbf{H}\mathbf{H}_2\mathbf{R} \tag{A16}$$

where $\mathbf{H}$ is the $n_a \times n_a$ Hankel matrix

$$\mathbf{H} = \begin{bmatrix} f(T+n_a-1), & f(T+n_a-2), & \dots, & f(T) \\ f(T+n_a-2), & f(T+n_a-3), & \dots, & f(T-1) \\ \vdots & \vdots & \ddots & \vdots \\ f(T), & f(T-1), & \dots, & f(T-n_a+1) \end{bmatrix}. \tag{A17}$$

Using arguments similar to those when $\mathbf{H}_2$ was analyzed, we claim that $\mathbf{H}$ is a nonsingular matrix. It follows from (A16) that if $\text{grad}_A J_{\text{PEIR}}(\mathbf{x}) = 0$, then $a_i = 0$ for $i = 1, \dots, n_a$, which means that $\mathbf{x} \in \mathcal{S}_0$.

Thesis (c): Sufficiency ($\Leftarrow$): If $\mathbf{x} \in \mathcal{S}_0$, then $\hat{F}(z) = 1$ and $\hat{F}_2(z) = 1$. Therefore, $f(t) = 0$ and $f_2(t) = 0$ for $t > 0$. By using (8), we conclude that grad $J_{\text{PEIR}} = 0$

Thesis (d): We use notation introduced in the proof of thesis (c). Note that $\|\text{grad}\, J_{\text{PEIR}}(\mathbf{x})\|_2 = \|\text{grad}_A J_{\text{PEIR}}(\mathbf{x})\|_2$. According to (A16) we obtain (A18), shown at the top of the page. Note that $\mathbf{H}\mathbf{H}_2\mathbf{R}$ is a nonsingular matrix. Therefore, $\mathbf{H}\mathbf{H}_2\mathbf{R}\mathbf{R}^{\text{T}}\mathbf{H}_2^{\text{T}}\mathbf{H}^{\text{T}}$ is a positive definite matrix. Moreover, $\|\mathbf{x} = \|[a_1, \quad a_2, \quad \dots, \quad a_{n_a}]\|_2$. Therefore, $\lim_{\|\mathbf{x}\| \to \infty} \|\text{grad} J_{\text{PEIR}}(\mathbf{x})\|_2 = \infty$.

Thesis (e): If $\mathbf{x} \in \mathcal{S}$, then both $\hat{F}(z)$ and $\hat{F}_2(z)$ have their poles inside $C(\rho)$. Hence, according to Lemma 1 $\lim_{t\to\infty} f(t) = 0$, and $\lim_{t\to\infty} f_2(t) = 0$. This observation along with (8) implies that $\lim_{T\to\infty} \|\text{grad}\, J_{\text{PEIR}}(\mathbf{x})\| = 0$.

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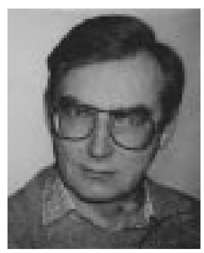

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